11. *Hertzer's* Table (pp. 249—251), giving the Residue-Index (q) of 10, when $q > 2$. [Hertzer's q is the ν of this Paper].

(1) *Insert the primes* (p) *marked*, and correct the Residue-indices* (q) *as follows*, [Argument p]:

$$p = 101009,\ {}^*106321,\ {}^*109873.$$
$$q = \ \ 16\ \ ,\ \ \ 4\ \ ,\ \ \ 7\ \ .$$

(2) *Transfer* the *ten* lines of Table at top of page 250 (without the heading) to foot of 249. Cancel the head-line on page 251, and transfer the sixteen lines of Table on page 251 to foot of page 250. The columns on (the amended) page 249 will then be *continuous*, and those on (the amended) page 250 will also be *continuous*. Complete the *first and last* lines of the Table on the amended pages 249, 250 as follows:

page	*line*	*p*	*q*	*p*	*q*	*p*	*q*	*p*	*q*	*p*	*q*
249	first	100109	29	101051	47	102013	4	102769	6	103903	3
249	last	101009	16	102001	200	102763	6	103889	4	104779	3
250	first	104789	17	106543	3	108301	3	109751	10	111253	4
250	last	106453	36	108223	3	109741	3	111229	13		

A NEW EXTENSION OF DIRICHLET'S THEOREM ON PRIME NUMBERS.

By *L. E. Dickson, Ph.D.*

After establishing his theorem on an arithmetical progression of integers, Dirichlet gave several generalizations, one to complex integers, another to properly primitive quadratic forms, and a third to a quadratic form and related linear form, the last two generalizations without complete proof. Proofs have been supplied by H. Weber, *Math. Annalen*, 20 (1882), pp. 301-329; A. Meyer, *Crelle*, 103 (1888), pp. 98–117; F. Mertens, *Wien. Ber.*, 104, pp. 1093–1153; 109, pp. 415–480; de la Vallée-Poussin, and others.

These theorems refer to a succession of simple primes. I wish to suggest the possibility of their extension* to successions of prime-pairs, triples, ..., m-tuples. The simplest problem relates to m linear forms $a_in + b_i$, where a_i and b_i are relatively prime integers, for $i = 1, \ldots, m$. Do there exist m linear

* Although numerous references to Dirichlet's theorems occur in *Fortschritte der Math.* 26–31, I find no indication of the proposed extension.

forms which give m prime numbers for the same value of n, where n runs through an infinite succession of positive integers?

We readily determine necessary conditions on such linear forms. For no value of n are $a_1 n + b_1$ and $a_2 n + b_2$ both prime numbers > 2 if a_1 and a_2 are odd, while b_1 and b_2 are such that one is even and the other odd. In fact, when these conditions hold, the forms may be written

$$(2q+1)\,n + 2r,\ \ (2s+1)\,n + 2t + 1.$$

But, for n even, the first is even; for n odd, the second is even. The proof of the general theorem is evident: For no value of n are $a_i n + b_i$ $(i = 1, \ldots, p)$ all prime numbers $> p$ if no a_i is divisible by the prime p, while $b_1/a_1, \ldots, b_p/a_p$ are congruent modulo p with $0, 1, \ldots, p-1$ in some order. Hence if we consider the m forms in sets of p, we obtain the

NECESSARY CONDITION. *In order that m forms $a_i n + b_i$ shall give m prime numbers for at least one integer n, it is necessary, for every prime $p \overline{\gtrless} m$ and for every set of p of the a_i chosen from those not divisible by p, that at least two of the b_i/a_i shall be congruent modulo p.*

The sufficiency of these conditions is proposed as a problem worthy of an investigation by one of the various number-theorists who have acquired such skill in the admirable treatment of Dirichlet's problems. As an incentive to this end, and for an application to simple groups, I give all the prime-triples in 64 sets of three linear forms $a_i n + b_i$ for $n = 0, 1, \ldots, 10$; and all in 3 of the sets for $n = 0, 1, \ldots, 100$. The results may also be interpreted as referring to 4 (or 8 or 16) sets of three forms for a wider range of values of n.

From the surprising frequency of the occurrence of prime-triples, it might be inferred that the forms were selected as favourable cases. They are, however, the only forms examined. From the standpoint of the question in hand, they may be said to have been chosen at random. Since in each set every a_i is divisible by 6, the above necessary conditions are satisfied.

For any prime $p > 3$, there is a simple group of order $\omega = \frac{1}{2} p\,(p^2 - 1)$. Now ω is divisible by 4 and not by 8 if and only if $p = 8k + 3$ or $8k + 5$. For $p = 8k + 5$, we may set $k = 3l + 1$ or $3l$ (since $k = 3l + 2$ makes p composite). For $p = 8k + 3$, we may set $k = 3l + 1$ or $3l + 2$ (since $p > 3$). We obtain for $\frac{1}{12}\omega$ the respective values:

$$(2l+1)\,(12l+7)\,(24l+13),\quad (4l+1)\,(6l+1)\,(24l+5),$$
$$(2l+1)\,(12l+5)\,(24l+11),\quad (4l+3)\,(6l+5)\,(24l+19).$$

We next assume that no one of the binomials is divisible by 3. Then $l = 3j$ or $3j + 2$ for the first, etc. The products then become

$$(6j+1)(36j+7)(72j+13), \quad (6j+5)(36j+31)(72j+61),$$

$$(12j+1)(18j+1)(72j+5), \quad (12j+5)(18j+7)(72j+29),$$

$$(6j+1)(36j+5)(72j+11), \quad (6j+5)(36j+29)(72j+59),$$

$$(12j+7)(18j+11)(72j+43), \quad (12j+11)(18j+17)(72j+67),$$

respectively. We next assume that no one of the new binomials is divisible by 5. Then $j = 5m$ or $5m + 2$ for the first, giving I. or II. of the table, respectively. The second for $j = 5m + 1$ or $5m + 3$ gives III. or IV. The third for $j = 5m + 1$ or $5m + 4$ gives V. or VI. The fourth for $j = 5m + 2$ or $5m + 4$ gives VII. or VIII. The fifth for $j = 5m + 1$ or $5m + 3$ gives IX. or X. The sixth for $j = 5m + 2$ or $5m + 4$ gives XI. or XII. The seventh for $j = 5m$ or $5m + 2$ gives XIII. or XIV. The eighth for $j = 5m$ or $5m + 3$ gives XV. or XVI. The subdivision of the 16 into 64 forms to exclude the factor 7 are made in the table.

Denote by $p_1, p_2, \ldots,$ the prime numbers > 3 in order. Then after r subdivisions of the above 8 sets, we obtain $N_r = 8(p_1 - 3)(p_2 - 3)\ldots(p_r - 3)$ sets of three forms. For every n each of the resulting forms is prime to $2, 3, p_1, \ldots, p_r$. The same prime-triple does not occur twice in the N_r sets. Hence if we could prove the existence of at least one prime-triple in each of the N_r sets for every value of r, we would have the theorem on the infinitude of prime-triples in the initial 4 sets of forms.

We conclude from the table that there are exactly 103 ways of expressing $\frac{1}{2}p(p^2 - 1)$ in the form $12pqr$, where p, q, r are prime numbers each $< 11.7.360 \equiv 27720$ and each > 7. Including the factorizations

$$5.7.29, \; 7.11.43, \; 5.29.59, \; 5.31.61, \; 7.41.83,$$

lost in passing from the 4 to the 64 sets of form, we obtain exactly 108 simple linear fractional groups of modulus $p < 27720$ whose orders are of the form $2^2.3.p.q.r$ where p, q, r are primes > 3. The table gives 29 further such orders, but of greater modulus.

Table of all the prime-triples in the 64 sets of three linear forms, for $n \overline{\leqq} 10$, and occasionally for a higher limit.

I. $30m+1$, $180m+7$, $360m+13$, $m=7n+1, 2, 4, 6$.

I_1. 661, 3967, 7933 ($n=3$).

I_2. 61, 367, 733 ($n=0$); 271, 1627, 3253 ($n=1$);
2161, 12967, 25933 ($n=10$).

I_4. 751, 4507, 9013 ($n=3$); 1381, 8287, 16573 ($n=6$).

I_6. 601, 3607, 7213 ($n=2$).

II. $30m+13$, $180m+79$, $360m+157$, $m=7n+0, 2, 3, 5$.

II_0. 13, 79, 157 ($n=0$); 1063, 6379, 12757 ($n=5$).

II_2. 73, 439, 877 ($n=0$).

II_3. 103, 619, 1237 ($n=0$); 1993, 11959, 23917 ($n=9$);
2203, 13219, 26437 ($n=10$).

II_5. 1423, 8539, 17077 ($n=6$).

III. $30m+11$, $180m+67$, $360m+133$, $m=7n+1, 3, 4, 6$.

III_1. 1091, 6547, 13093 ($n=5$); 1511, 9067, 18133 ($n=7$);
1931, 11587, 23173 ($n=9$).

III_3. 101, 607, 1213 ($n=0$); 311, 1867, 3733 ($n=1$);
1361, 8167, 16333 ($n=6$).

III_4. 761, 4567, 9133 ($n=3$).

III_6. 2711, 16267, 32533 ($n=12$).

IV. $30m+23$, $180m+139$, $360m+277$, $m=7n+0, 2, 4, 5$.

IV_0. 23, 139, 277 ($n=0$); 233, 1399, 2797 ($n=1$);
863, 5179, 10357 ($n=4$).

IV_2. 83, 499, 997 ($n=0$); 293, 1759, 3517 ($n=1$);
503, 3019, 6037 ($n=2$); 1553, 9319, 18637 ($n=7$);
1973, 11839, 23677 ($n=9$);
4493, 26959, 53917 ($n=21$);
4703, 28219, 56437 ($n=22$);
6173, 37039, 74077 ($n=29$);
6803, 40819, 81637 ($n=32$);
8273, 49639, 99277 ($n=39$);
14153, 84919, 169837 ($n=67$);
15413, 92479, 184957 ($n=73$);
20663, 123979, 247957 ($n=98$).

IV_4. 773, 4639, 9277 ($n=3$).

IV_5. 1013, 6079, 12157 ($n=4$).

V. $60m+13$, $90m+19$, $360m+77$, $m=7n+1, 3, 4, 6$.

V_1. 3853, 5779, 23117 ($n=9$).

V_3. 613, 919, 3677 ($n=1$); 1033, 1549, 6197 ($n=2$).

V_4. 6553, 9829, 39317 ($n=15$);
11593, 17389, 69557 ($n=27$);
15373, 23059, 92237 ($n=36$);
17053, 25579, 102317 ($n=40$);
25033, 37549, 150197 ($n=59$);
41413, 62119, 248477 ($n=98$);

V_6. 4153, 6229, 24917 ($n=9$).

VI. $60m+49$, $90m+73$, $360m+293$, $m=7n+1, 2, 4, 6$.

VI_1. 109, 163, 653 ($n=0$); 1789, 2683, 10733 ($n=4$).

VI_2. 1429, 2143, 8573 ($n=3$).

VI_4. 709, 1063, 4253 ($n=1$).

VI_6. 3769, 5653, 22613 ($n=8$).

VII. $60m+29$, $90m+43$, $360m+173$, $m=7n+0, 2, 4, 6$.

VII_0. 29, 43, 173 ($n=0$); 449, 673, 2693 ($n=1$);
4229, 6343, 25373 ($n=10$).

VII_2. 569, 853, 3413 ($n=1$).

VII_4. 1109, 1663, 6653 ($n=2$).

VII_6. 3329, 4993, 19973 ($n=7$).

VIII. $60m+53$, $90m+79$, $360m+317$, $m=7n+0, 1, 3, 5$.

VIII_0. 53, 79, 317 ($n=0$); 3413, 5119, 20477 ($n=8$).

VIII_1. 953, 1429, 5717 ($n=2$).

VIII_3. 2333, 3499, 13997 ($n=5$).

VIII_5. 4973, 7459, 29837 ($n=11$);
5813, 8719, 34877 ($n=13$).

IX. $30m+7$, $180m+41$, $360m+83$, $m=7n+1, 2, 4, 6$.

IX_1. 457, 2741, 5483 ($n=2$); 1087, 6521, 13043 ($n=5$);
2137, 12821, 25643 ($n=10$).

IX_2. 907, 5441, 10883 ($n = 4$).

IX_4. 127, 761, 1523 ($n = 0$); 2017, 12101, 24203 ($n = 9$).

IX_6. 4597, 27581, 55163 ($n = 21$);
8377, 50261, 100523 ($n = 39$);
15307, 91841, 183683 ($n = 72$);
16987, 101921, 203843 ($n = 80$);
19507, 117041, 234083 ($n = 92$).

X. $30m+19$, $180m+113$, $360m+227$, $m = 7n+0, 2, 3, 5$.

X_0. 19, 113, 227 ($n = 0$).

X_2. 1549, 9293, 18587 ($n = 7$).

X_3. 109, 653, 1307 ($n = 0$); 1579, 9473, 18947 ($n = 7$);
1789, 10733, 21467 ($n = 8$).

X_5. 379, 2273, 4547 ($n = 1$); 1009, 6053, 12107 ($n = 4$).

XI. $30m+17$, $180m+101$, $360m+203$, $m = 7n+1, 3, 4, 6$.

XI_1. 47, 281, 563 ($n = 0$); 1097, 6581, 13163 ($n = 5$);
1307, 7841, 15683 ($n = 6$).

XI_3. 107, 641, 1283 ($n = 0$); 317, 1901, 3803 ($n = 1$).

XI_4. 1187, 7121, 14243 ($n = 5$).

XI_6. 3557, 21341, 42683 ($n = 16$).

XII. $30m+29$, $180m+173$, $360m+347$, $m = 7n+0, 2, 4, 5$.

XII_0. 29, 173, 347 ($n = 0$); 449, 2693, 5387 ($n = 2$);
1709, 10253, 20507 ($n = 8$).

XII_2. 3449, 20693, 41387 ($n = 16$);
4079, 24473, 48947 ($n = 19$).

XII_4. 569, 3413, 6827 ($n = 2$).

XII_5. 599, 3593, 7187 ($n = 2$); 1019, 6113, 12227 ($n = 4$).

XIII. $60m+7$, $90m+11$, $360m+43$, $m = 7n+1, 3, 5, 6$.

XIII_1. 907, 1361, 5443 ($n = 2$).

XIII_3. 607, 911, 3643 ($n - 1$).

XIII_5. 727, 1091, 4363 ($n=1$); 1567, 2351, 9403 ($n=3$); 4507, 6761, 27043 ($n=10$).

XIII_6. 787, 1181, 4723 ($n=1$).

XIV. $60m+31$, $90m+47$, $360m+187$, $m=7n+0, 2, 4, 6$.

XIV_0. 2971, 4457, 17827 ($n=7$); 3391, 5087, 20347 ($n=8$).

XIV_2. 151, 227, 907 ($n=0$).

XIV_4. 1531, 2297, 9187 ($n=3$).

XIV_6. 5431, 8147, 32587 ($n=12$); 9631, 14447, 57787 ($n=22$).

XV. $60m+11$, $90m+17$, $360m+67$, $m=7n+0, 2, 4, 5$.

XV_0. 11, 17, 67 ($n=0$); 2531, 3797, 15187 ($n=6$).

XV_2. 131, 197, 787 ($n=0$); 3491, 5237, 20947 ($n=8$).

XV_4. 1091, 1637, 6547 ($n=2$); 1511, 2267, 9067 ($n=3$); 1931, 2897, 11587 ($n=4$); 2351, 3527, 14107 ($n=5$).

XV_5. 311, 467, 1867 ($n=0$); 3251, 4877, 19507 ($n=7$); 3671, 5507, 22027 ($n=8$).

XVI. $60m+47$, $90m+71$, $360m+283$, $m=7n+0, 2, 3, 5$.

XVI_0. 47, 71, 283 ($n=0$); 467, 701, 2803 ($n=1$).

XVI_2. 11087, 16631, 66523 ($n=26$); 11927, 17891, 71563 ($n=28$).

XVI_3. 1907, 2861, 11443 ($n=4$); 4007, 6011, 24043 ($n=9$).

XVI_5. 347, 521, 2083 ($n=0$); 2027, 3041, 12163 ($n=4$); 2447, 3671, 14683 ($n=5$).

The run of four in XV_4 is noteworthy. As to the accuracy of the work, in which suitable checks were employed, it may be noted that, on subsequently comparing the numbers with a table of primes, every number was found to be prime without exception.

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